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Construction of C^*-algebras

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1 Crossed products

In this note, we discuss several generalizations of dynamical systems and their crossed products. Throughout this note, A denotes a C^*-algebra.

Let G be a locally compact group. An *action* of G on A is a strongly continuous homomorphism $\alpha\colon G\to \operatorname{Aut}(A)$. The triple (A, G, α) is called a *C^*-dynamical system*. From a C^*-dynamical system (A, G, α), we get a C^*-algebra $A\rtimes_{\alpha}G$ which is called the *crossed product*† (see [Pe], for the detail).

When $G=\mathbb{Z}$, an action $\alpha\colon \mathbb{Z}\to \operatorname{Aut}(A)$ is determined by $\alpha_1\in \operatorname{Aut}(A)$. By an abuse of notation, we denote α_1 by α, and identify actions of $\mathbb{Z}$ and automorphisms. The C^*-algebra $A\rtimes_{\alpha}\mathbb{Z}$ is sometimes called the *crossed product by the automorphism* α.

Definition 1.1 The crossed product $A\rtimes_{\alpha}\mathbb{Z}$ is the universal C^*-algebra generated by the images of the $*$-homomorphism $\pi\colon A\to A\rtimes_{\alpha}\mathbb{Z}$ and the linear map $t\colon A\to A\rtimes_{\alpha}\mathbb{Z}$ satisfying

(i) $t(x)\pi(a)=t(xa)$,

(ii) $t(x)^*t(y)=\pi(x^*y)$,

(iii) $\pi(a)t(x)=t(\alpha(a)x)$,

(iv) $t(x)t(y)^*=\pi(\alpha^{-1}(xy^*))$

for $a, x, y\in A$.

In the definition above, "universal" means that for any C^*-algebra B, any $*$-homomorphism $\pi'\colon A\to B$ and any linear map $t'\colon A\to B$ satisfying (i) – (iv) above, there exists a $*$-homomorphism $\rho\colon A\rtimes_{\alpha}\mathbb{Z}\to B$ such that $\pi'=\rho\circ\pi$ and $t'=\rho\circ t$. We can show that there exists a unitary u in the multiplier algebra of $A\rtimes_{\alpha}\mathbb{Z}$ such that $t(x)=u\pi(x)$ for $x\in A$. This unitary u satisfies

$$u\pi(a)u^*=\pi(\alpha^{-1}(a))\qquad \text{for } a\in A. \tag{$*$}$$

†There are two types of crossed products, namely the reduced ones and the full ones. We do not go to the detail because we are only interested in the case $G=\mathbb{Z}$ where the two types of C^*-algebras coincide.

Conversely, if a $*$-homomorphism $\pi' \colon A \to B$ and a unitary u' in the multiplier algebra of B satisfies $(*)$, then the pair of the $*$-homomorphism pi' and the linear map $t' \colon A \to B$ defined by $t(x) = u\pi(x)$ for $x \in A$ satisfies (i) – (iv). Thus the above definition coincides with the ordinal one using the covariant condition $(*)$ (see for example [Pe]). There are many generalizations of this construction. One of them is a crossed product by a Hilbert C^*-bimodule [AEE].

Definition 1.2 **([BMS])** A *Hilbert A-bimodule* X is a Banach space which is an A-bimodule and has A-valued left and right inner products $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ such that

(i) $(\xi,\xi) \geq 0, \quad \langle\xi,\xi\rangle \geq 0,$

(ii) $\|\xi\| = \|(\xi,\xi)\|^{1/2} = \|\langle\xi,\xi\rangle\|^{1/2},$

(iii) $(a\xi,\eta) = a(\xi,\eta), \quad \langle\xi,\eta a\rangle = \langle\xi,\eta\rangle a,$

(iv) $(\xi,\eta)\zeta = \xi\langle\eta,\zeta\rangle$

for $\xi,\eta,\zeta \in X$, $a \in A$.

For $\xi,\eta \in X$ and $a \in A$, we can show $(\eta,\xi) = (\xi,\eta)^*, \langle\eta,\xi\rangle = \langle\xi,\eta\rangle^*$ from (i), and

$$(\xi a,\eta) = (\xi,\eta a^*), \quad \langle\xi,a\eta\rangle = \langle a^*\xi,\eta\rangle$$

from (iv). An automorphism $\alpha \in \mathrm{Aut}(A)$ determines a Hilbert A-bimodule X_α as follows: As Banach spaces, X_α is isomorphic to A via the map $A \ni x \mapsto \xi_x \in X_\alpha$. The bimodule structure and inner products are defined as

$$a\xi_x b := \xi_{\alpha(a)xb}, \quad (\xi_x,\xi_y) := \alpha^{-1}(xy^*), \quad \langle\xi_x,\xi_y\rangle := x^*y$$

for $a,x,y \in A$. By this construction, we think that Hilbert C^*-bimodules generalize automorphisms. The compositions of automorphisms correspond to the tensor products of Hilbert C^*-bimodules[†], and the inverses correspond to the dual Hilbert C^*-bimodules.

Definition 1.3 **([AEE, Definition 2.1])** The *crossed product* $A \rtimes_X \mathbb{Z}$ of a C^*-algebra A by a Hilbert A-bimodule X is the universal C^*-algebra generated by the images of the $*$-homomorphism $\pi \colon A \to A \rtimes_X \mathbb{Z}$ and the linear map $t \colon X \to A \rtimes_X \mathbb{Z}$ satisfying

(i) $t(\xi)\pi(a) = t(\xi a),$

(ii) $t(\xi)^*t(\eta) = \pi(\langle\xi,\eta\rangle),$

(iii) $\pi(a)t(\xi) = t(a\xi),$

(iv) $t(\xi)t(\eta)^* = \pi((\xi,\eta)),$

for $a \in A$ and $\xi,\eta \in X$.

[†]With our convention, we have $X_\alpha \otimes X_\beta \cong X_{\beta\circ\alpha}$.

The conditions (i) and (iii) hold automatically from the conditions (ii) and (iv), respectively. It is straightforward to see $A \rtimes_{X_\alpha} \mathbb{Z} \cong A \rtimes_\alpha \mathbb{Z}$ for $\alpha \in \mathrm{Aut}(A)$.

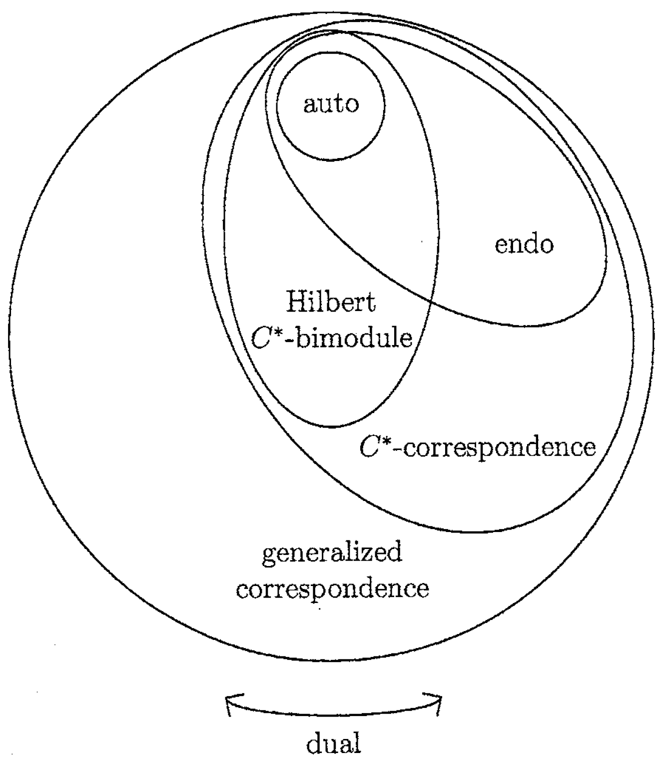
Another generalization of the crossed products by automorphisms is crossed products by endomorphisms [M, St]. These two generalizations can be unified to the construction of the *Pimsner algebra* $\mathcal{O}_X$ from a C^*-correspondence† X, which is defined in [Pi] and modified in [Ka5].

Definition 1.4 If a Banach space X satisfies all the conditions for Hilbert A-bimodules except the existence of a left inner product but instead satisfies $\langle a\xi, \eta\rangle = \langle \xi, a^*\eta\rangle$ for $\xi, \eta \in X$ and $a \in A$, then it is called a *C^*-correspondence* over A.

For a definition and properties of the Pimsner algebra, see the next section. Recently, Exel defines generalized correspondences and gives a method to construct C^*-algebras from them ([E]). A *ternary ring of operators* (TRO) is a Banach space X with a *ternary operation* $[\cdot,\cdot,\cdot]\colon X \times X \times X \to X$ which satisfies the conditions that the map $(x, y, z) \mapsto xy^*z$ satisfies ([Z]). A *generalized correspondence* over A is an A-bimodule which is a TRO such that the ternary operation satisfies

$$[\xi, a\eta, \zeta] = [\xi, \eta, a^*\zeta], \qquad [\xi, \eta a, \zeta] = [\xi a^*, \eta, \zeta]$$

for $\xi, \eta, \zeta \in X$ and $a \in A$. A C^*-correspondence is a generalized correspondence by setting $[\xi, \eta, \zeta] := \xi\langle \eta, \zeta\rangle$.



†Pimsner called it a Hilbert-bimodule, and he assumed that its left action is faithful.

The class of generalized correspondences is a natural class which contains C^*-correspondences and is invariant under "taking duals". In [E], Exel suggests one way to construct a C^*-algebra $C^*(A, X)$ from a generalized correspondence X over A, which generalizes the construction of Pimsner algebras. There are several things remained which have to be checked. For example, we do not know whether the natural embedding map $A \to C^*(A, X)$ is injective or not.

So far, we only consider the generalization of actions and crossed products for the case that the group is $\mathbb{Z}$ (or the semigroup $\mathbb{N}$). There is a generalization of actions by general groups using C^*-correspondences, which is called a *product system.*

Definition 1.5 *Let Γ be a cone of a group. A* product system *over Γ is a family $\{X_\gamma\}_{\gamma\in\Gamma}$ of C^*-correspondences over A together with the isomorphisms as C^*-correspondences*

$$w_{\gamma,\mu}\colon X_\gamma \otimes X_\mu \to X_{\gamma\mu},$$

satisfying the associative low

$$w_{\gamma\mu,\nu} \circ (w_{\gamma,\mu} \otimes \mathrm{id}_{X_\nu}) = w_{\gamma,\mu\nu} \circ (\mathrm{id}_{X_\gamma} \otimes w_{\mu,\nu}).$$

We should be careful of X_e where $e \in \Gamma$ is the identity (see [F]). If Γ has a topology (e.g. $\Gamma = \mathbb{R}_+$), then we have to take care of the "continuity" (or "measurability") of the map $\gamma \to X_\gamma$ (see [H]). Product systems over the positive real line $\mathbb{R}_+$ are related to E_0-semigroup (see [H, Sk]). A *higher rank graph* introduced in [KP] gives an example of product systems over the semigroup $\mathbb{N}^k$ (see [F, RSY]).

There is a natural construction of a C^*-algebra from a product system, which is analogue of Toeplitz algebra $\mathcal{T}_X$ defined below. However, except for special cases, we do not know how to define analogues of crossed products or Pimsner algebras of product systems.

2 Pimsner algebras

Let A be a C^*-algebra, and X be a C^*-correspondence over A.

Definition 2.1 A *representation* of X on a C^*-algebra B is a pair (π, t) consisting of a $*$-homomorphism $\pi\colon A \to B$ and a linear map $t\colon X \to B$ satisfying

(i) $t(\xi)\pi(a) = t(\xi a)$,

(ii) $t(\xi)^* t(\eta) = \pi(\langle \xi, \eta\rangle)$,

(iii) $\pi(a)t(\xi) = t(a\xi)$

for $a \in A$ and $\xi, \eta \in X$. We denote by $C^*(\pi, t)$ the C^*-algebra generated by the images of π and t in B.

Definition 2.2 We denote the universal representation by $(\bar{\pi}_X, \bar{t}_X)$. The C^*-algebra $C^*(\bar{\pi}_X, \bar{t}_X)$ is called the *Toeplitz algebra* of X, and denoted by $\mathcal{T}_X$.

The Toeplitz algebra $\mathcal{T}_X$ is not an analogue of crossed products. We need the condition corresponding (iv) in Definition 1.1 or Definition 1.3. To express this condition, we introduce some notations.

Definition 2.3 A map $T\colon X \to X$ is said to be *adjointable* if there exists $T^*\colon X \to X$ such that $\langle \xi, T\eta\rangle = \langle T^*\xi, \eta\rangle$ for $\xi, \eta \in X$.

We denote by $\mathcal{L}(X)$ the set of all adjointable operators on X.

It is routine to check that $\mathcal{L}(X)$ is a C^*-algebra, and the left action defines the $*$-homomorphism $\varphi\colon A \to \mathcal{L}(X)$ by $\varphi(a)\xi = a\xi$.

Definition 2.4 For $\xi, \eta \in X$, the operator $\theta_{\xi,\eta} \in \mathcal{L}(X)$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta\rangle$ for $\zeta \in X$. We define $\mathcal{K}(X) \subset \mathcal{L}(X)$ by

$$\mathcal{K}(X) = \overline{\operatorname{span}}\{\theta_{\xi,\eta} \mid \xi, \eta \in X\},$$

which is an ideal of $\mathcal{L}(X)$.

For the proof of the next lemma see [KPW, Lemma 2.2] or [FR, Remark 1.7].

Lemma 2.5 *For a representation (π, t) of X, there exists a unique $*$-homomorphism $\psi_t\colon \mathcal{K}(X) \to C^*(\pi, t)$ such that $\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$ for $\xi, \eta \in X$.*

Definition 2.6 For a C^*-correspondence X, we define an ideal J_X of A by

$$J_X := \{a \in A \mid \varphi(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \ker\varphi\}.$$

Definition 2.7 A representation (π, t) of X is said to be *covariant* if $\psi_t(\varphi(a)) = \pi(a)$ for all $a \in J_X$.

Definition 2.8 Let (π_X, t_X) be the universal covariant representation, and set $\mathcal{O}_X := C^*(\pi_X, t_X)$ which is called the *Pimsner algebra* of X.

One can check that this construction generalizes the crossed products by endomorphisms and the ones by Hilbert C^*-bimodules as well as other classes of C^*-algebras (see Section 3). We will give several characterizations of the representation (π_X, t_X) and the Pimsner algebra $\mathcal{O}_X$.

Definition 2.9 For two representations (π_1, t_1) and (π_2, t_2) of X, we write $(\pi_1, t_1) \succeq (\pi_2, t_2)$ if there exists a $*$-homomorphism $\rho\colon C^*(\pi_1, t_1) \to C^*(\pi_2, t_2)$ such that $\pi_2 = \rho \circ \pi_1$ and $t_2 = \rho \circ t_1$.

Such a $*$-homomorphism ρ is, if it exists, unique and surjective. We will say that two representations (π_1, t_1) and (π_2, t_2) are *equivalent* if $(\pi_1, t_1) \succeq (\pi_2, t_2)$ and $(\pi_1, t_1) \preceq (\pi_2, t_2)$. This is the same as the existence of an isomorphism $\rho\colon C^*(\pi_1, t_1) \to C^*(\pi_2, t_2)$ with $\pi_2 = \rho \circ \pi_1$ and $t_2 = \rho \circ t_1$. The set of equivalence classes of representations is an ordered set by the order $\preceq$. The universal representation $(\bar{\pi}_X, \bar{t}_X)$ is the largest element in this set.

Definition 2.10 A representation (π, t) of X is said to be *injective* if a $*$-homomorphism π is injective, and said to *admit a gauge action* if for each $z \in \mathbb{T}$, there exists a $*$-homomorphism $\beta_z\colon C^*(\pi,t) \to C^*(\pi,t)$ such that $\beta_z(\pi(a)) = \pi(a)$ and $\beta_z(t(\xi)) = zt(\xi)$ for all $a \in A$ and $\xi \in X$.

By the universality, the representation (π_X, t_X) on $\mathcal{O}_X$ admits a gauge action. We denote this action by $\gamma\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_X)$ and call it the *gauge action* on $\mathcal{O}_X$. We can also see that (π_X, t_X) is injective by using Fock representation [Ka6].

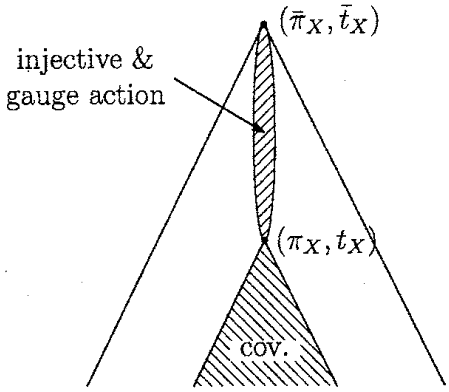
Theorem 2.11 ([Ka6, Theorem 6.4], [Ka7, Propostion 7.14]) *Each of the following three conditions characterizes the representation (π_X, t_X) on the Pimsner algebra $\mathcal{O}_X$:*

(i) *(π_X, t_X) is the largest in the set of all covariant representations.*

(ii) *(π_X, t_X) is the smallest in the set of all injective representations admitting gauge actions.*

(iii) *(π_X, t_X) is the only injective covariant representation admitting a gauge action.*

(i) is nothing but the definition of (π_X, t_X). The uniqueness part of (iii) is called the *gauge-invariant uniqueness theorem.* (ii) gives characterizations of (π_X, t_X) and $\mathcal{O}_X$ without using the covariance nor the ideal J_X.



The most important part of the proof of Theorem 2.11 is an analysis of the fixed point algebra $\mathcal{O}_X^\gamma$ of the gauge action (see the proof of the next theorem).

Theorem 2.12 (see [DS, Theorem 3.1], [Ka6, Theorems 7.1, 7.2])
A: nuclear $\Rightarrow$ $\mathcal{O}_X^\gamma$: nuclear $\iff$ $\mathcal{O}_X$: nuclear.
A: exact $\iff$ $\mathcal{O}_X^\gamma$: exact $\iff$ $\mathcal{O}_X$: exact.

Sketch of Proof. The two equivalences

"$\mathcal{O}_X^\gamma$: nuclear $\iff$ $\mathcal{O}_X$: nuclear", "$\mathcal{O}_X^\gamma$: exact $\iff$ $\mathcal{O}_X$: exact" follow from the general fact on fixed point algebras by actions of compact groups (see [DLRZ]). We sketch the proof of "A: nuclear $\Rightarrow \mathcal{O}_X^\gamma$: nuclear" (the corresponding statement for exactness can be proven similarly).

Suppose that A is nuclear, and we will prove that $\mathcal{O}_X^\gamma$ is nuclear. We set $Y_0 = \pi_X(A) \subset \mathcal{O}_X$ and

$$Y_{n+1} = t_X(X)Y_n := \overline{\operatorname{span}}\{xy \in \mathcal{O}_X \mid x \in t_X(X),\ y \in Y_n\}$$

for $n \in \mathbb{N}$. Then we have

$$\mathcal{O}_X = \overline{\operatorname{span}}\Big(\bigcup_{n,m\in\mathbb{N}} Y_nY_m^*\Big), \quad \mathcal{O}_X^\gamma = \overline{\operatorname{span}}\Big(\bigcup_{n\in\mathbb{N}} Y_nY_n^*\Big).$$

We set $B_n = Y_nY_n^*$ and $B_{[0,n]} = B_0 + B_1 + \cdots + B_n$. Then we have $\mathcal{O}_X^\gamma = \varinjlim B_{[0,n]}$. It suffices to show that the C^*-algebra $B_{[0,n]}$ is nuclear for all $n \in \mathbb{N}$. We will prove this by induction on n. The C^*-algebra $B_{[0,0]} = B_0 \cong A$ is nuclear by the assumption. Suppose we will prove that $B_{[0,n-1]}$ is nuclear. The C^*-algebra B_n is strongly Morita equivalent to the C^*-algebra $Y_n^*Y_n \subset \mathcal{O}_X$ which is isomorphic to an ideal of A. Hence B_n is nuclear. Since B_n is an ideal of $B_{[0,n]}$ and $B_{[0,n]} = B_{[0,n-1]} + B_n$, we have $B_{[0,n]}/B_n \cong B_{[0,n-1]}/(B_{[0,n-1]} \cap B_n)$ which is nuclear.

$$\begin{tikzcd}
0 \arrow[r] & B_{[0,n-1]} \cap B_n \arrow[r] \arrow[d] & B_{[0,n-1]} \arrow[r] \arrow[d] & B_{[0,n-1]}/(B_{[0,n-1]} \cap B_n) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B_n \arrow[r] & B_{[0,n]} \arrow[r] & B_{[0,n]}/B_n \arrow[r] & 0
\end{tikzcd}$$

Therefore $B_{[0,n]}$ is nuclear being an extension of nuclear C^*-algebras. This completes the proof. ∎

Remark 2.13 $\mathcal{T}_X$ is nuclear (resp. exact) if and only if A is nuclear (resp. exact). There is an example of a C^*-correspondence X over a non-nuclear C^*-algebra A such that $\mathcal{O}_X$ is nuclear (see [Ka6, Example 7.7]).

There have been some results on the ideal structures of Pimsner algebras ([Ka7], [MT1]), and a criterion for their simplicity in a special case ([Sc]). However we do not know when they are simple in general. On the K-theory of Pimsner algebras, we have the following (see [Pi, Theorem 4.9] and [Ka6, Theorem 8.6, Proposition 8.8]).

Theorem 2.14 *The Pimsner algebra $\mathcal{O}_X$ satisfies the Universal Coefficient Theorem of [RS], if both A and J_X satisfy it. We have the following exact sequence;*

$$\begin{tikzcd}
K_0(J_X) \arrow[r, "\iota_*-[X]"'] & K_0(A) \arrow[r, "(\pi_X)_*"'] & K_0(\mathcal{O}_X) \arrow[d] \\
K_1(\mathcal{O}_X) \arrow[u] & K_1(A) \arrow[l, "(\pi_X)_*"'] & K_1(J_X). \arrow[l, "\iota_*-[X]"']
\end{tikzcd}$$

3 Topological quivers

In this section, we give methods to construct C^*-correspondences over commutative C^*-algebras.

Definition 3.1 ([MT2]) A *topological quiver* $\mathcal{Q} = (E^0, E^1, d, r, \lambda)$ consists of two locally compact spaces E^0 and E^1, a continuous open map $d\colon E^1 \to E^0$, a continuous map $r\colon E^1 \to E^0$, and a family of Radon measures $\lambda = \{\lambda_v\}_{v \in E^0}$ on E^1 satisfying the following two conditions:

(i) $\operatorname{supp} \lambda_v = d^{-1}(v)$ for all $v \in E^0$,

(ii) $v \mapsto \int_{E^1} \xi(e) d\lambda_v(e)$ is an element of $C_c(E^0)$ for all $\xi \in C_c(E^1)$.

Take a topological quiver $\mathcal{Q} = (E^0, E^1, d, r, \lambda)$. We set $A := C_0(E^0)$. For $\xi, \eta \in C_c(E^1)$,

$$v \mapsto \int_{E^1} \overline{\xi(e)} \eta(e) d\lambda_v(e)$$

is an element of $C_c(E^0)$. We denote this function by $\langle \xi, \eta \rangle \in A$. The linear space $C_c(E^1)$ is an A-bimodule by

$$f\xi g\colon E^1 \ni e \mapsto f(r(e))\xi(e)g(d(e))$$

for $f, g \in A$ and $\xi \in C_c(E^1)$. Let X be the completion of $C_c(E^0)$ with respect to the norm defined by $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$. The A-valued inner product and the A-bimodule structure are naturally extended to X. Thus X is a C^*-correspondence over A.

Definition 3.2 The Pimsner algebra $\mathcal{O}_X$ of the C^*-correspondence X over A constructed above is said to be the *C^*-algebra associated to $\mathcal{Q}$*, and denoted by $C^*(\mathcal{Q})$.

A quadruple $E = (E^0, E^1, d, r)$ consisting of two locally compact spaces E^0 and E^1, a local homeomorphism $d\colon E^1 \to E^0$, and a continuous map $r\colon E^1 \to E^0$, is called a *topological graph* ([Ka1]). For a topological graph $E = (E^0, E^1, d, r)$, the quintuple $\mathcal{Q}_E = (E^0, E^1, d, r, \lambda)$ is a topological quiver, where λ_v is the counting measures on $d^{-1}(v)$ for $v \in E^0$. The C^*-algebra $C^*(\mathcal{Q}_E)$ is denoted by $\mathcal{O}(E)$ in [Ka1]. When $d\colon E^1 \to E^0$ is a branched covering between Riemann surfaces, the counting measures λ_v on $d^{-1}(v)$ for $v \in E^0$ with multiplicities at branched points satisfy two conditions in Definition 3.1. Thus we get a topological quiver, and the C^*-algebras associated to this type of topological quivers are analyzed in [KW].

For C^*-algebras associated to topological quivers, we know the conditions for the simplicity ([MT2, Theorem 10.2], see also [Ka3, Theorem 8.12]).

By Theorems 2.12 and 2.14, the class of the C^*-algebras associated to topological quivers are included in the class of nuclear C^*-algebras satisfying the Universal Coefficient Theorem. There may be possibilities that all separable simple nuclear C^*-algebras satisfying the Universal Coefficient Theorem can be obtained as C^*-algebras associated to topological quivers. In fact, the following C^*-algebras were shown to be obtained as C^*-algebras associated to topological quivers (or actually topological graphs [Ka2, Ka4]):

(i) all AF-algebras,

(ii) many ASH-algebras including all simple AT-algebras with real rank zero,

(iii) all classifiable Kirchberg algebras.

We do not know whether the following examples arise as C^*-algebras associated to topological quivers:

(i) a simple C^*-algebra with a finite and an infinite projection found in [Ro],

(ii) all TAF-algebras classified in [L],

(iii) the Jiang and Su algebra $\mathcal{Z}$ defined in [JS].

A dynamical system $(C_0(\Omega), G, \alpha)$ of a commutative C^*-algebra $C_0(\Omega)$ gives rise to an action of G on the space Ω. Such an action defines a groupoid $\Omega \rtimes G$ which is called a *transformation group*, and the crossed product $C_0(\Omega) \rtimes_\alpha G$ is isomorphic to the C^*-algebra of this groupoid [Re]. From a topological graph E, we can construct a groupoid $\mathcal{G}_E$ using negative orbits so that the C^*-algebra $\mathcal{O}(E)$ is isomorphic to the C^*-algebra of the groupoid $\mathcal{G}_E$. This observation may help when we try to extend the construction in this section to the more general setting involving general groups.

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